ON RULED SURFACES WHOSE FLECNODE CURVE INTERSECTS EVERY GENERATOR IN TWO COINCIDENT POINTS*

BY

E. J. WILCZYNSKI†

The formulæ and the theorems developed in my recent paper *Studies in the general theory of ruled surfaces* are not directly applicable to the case when $\theta_4 = 0$, i. e., when the flecnode curve intersects every generator in two coincident points. The general notions, employed in that paper,‡ may however be applied to this special case as well, and give rise to a number of interesting and important considerations.

§ 1. *The covariant C_3.*

The interpretation of the covariant C_3 given in *Covariants* and *Studies* is complete for the case $\theta_4 \neq 0$. It breaks down absolutely for $\theta_4 = 0$.

This covariant is

$$(1) \qquad\qquad C_3 = \alpha z - \beta y,$$

where

$$(2) \qquad \begin{aligned} \alpha &= 2(u_{11} - u_{22})\rho + 4u_{12}\sigma + \tfrac{1}{2}(v_{11} - v_{22})y + v_{12}z, \\ \beta &= 4u_{21}\rho - 2(u_{11} - u_{22})\sigma + v_{21}y - \tfrac{1}{2}(v_{11} - v_{22})z.\,§ \end{aligned}$$

We have

$$\theta_4 = (u_{11} - u_{22})^2 + 4u_{12}u_{21} = 0.$$

Let us assume that the curve C_y is the flecnode curve, so that $u_{12} = 0$. We shall then have also $u_{11} - u_{22} = 0$. Further we may assume $p_{11} = p_{22} = 0$.

We have therefore

$$u_{12} = u_{11} - u_{22} = 0, \qquad p_{11} = p_{22} = 0,$$

whence

$$(3) \qquad\qquad \alpha = p_{12}u_{21}y, \qquad \beta = 4u_{21}\rho + v_{21}y - p_{12}u_{21}z.$$

If a transformation of the independent variable be made by putting $\xi = \xi_{(x)}$, we find that for the new system of differential equations

$$\bar{v}_{21} = \frac{1}{(\xi')^3}(v_{21} - 4u_{21}\eta), \qquad \eta = \frac{\xi''}{\xi'}.^*$$

Therefore if $u_{21} \neq 0$, i. e., if S is not a quadric, we can always choose η in just one way so as to make $\bar{v}_{21} = 0$. We obtain therefore a perfectly definite surface of the congruence Γ, which we will call its *principal* surface, and which we shall characterize geometrically farther on.

Let us assume that the variable x has already been chosen in such a way that S', the derivative of S with respect to x, shall coincide with the principal surface of Γ. Then $v_{21} = 0$, and (3) becomes

$$\alpha = p_{12}u_{21}y, \qquad \beta = 4u_{21}\rho - p_{12}u_{21}z = u_{21}\bar{\beta},$$

where

$$\bar{\beta} = 4\rho - p_{12}z.$$

We have further in general (*Covariants*, p. 430),

$$\rho = 2y' + p_{11}y + p_{12}z,$$

whence

$$2y' = \rho - p_{12}z.$$

The point, whose coördinates are $y_1', \cdots, y_4'$, is therefore obviously the intersection of the tangent to the flecnode curve with the line $P_z P_\rho$. The point P_β whose coördinates are given by $\beta_1 \cdots \beta_4$ is also on the line $P_z P_\rho$, and the cross ratio of the four points $P_\rho P_\beta P_z P_{y'}$ is $\frac{1}{4}$. The point P_α obviously coincides with P_y.

The ruled surface which the covariant C_3 adjoins to S may therefore be defined as follows. In the plane tangent to S at its flecnode P_y, construct a line passing through P_y such that it, together with the generator, the flecnode tangent and the tangent of the flecnode curve shall constitute a plane pencil whose anharmonic ratio is $\frac{1}{4}$. The locus of these lines is the required ruled surface. Moreover the points of any generator of this surface are, by means of the covariant C_3 put into a one-to-one correspondence with those of g. The lines joining corresponding points pass through P_ρ, that point of the principal surface of the congruence Γ which corresponds to P_y.

It only remains to give a characteristic geometric property of the principal surface of Γ. For this purpose let us assume, in addition to our previous hypotheses, that C_z is an asymptotic curve on S, i. e., let $p_{21} = 0$. Then (*Congruence*, p. 188).

$$2\sigma' = u_{21}y + u_{11}z,$$

(4)

$$2\sigma'' = u'_{21}y + (u'_{11} - \tfrac{1}{2}u_{21}p_{12})z + \tfrac{1}{2}(u_{21}\rho + u_{11}\sigma),$$

and

$$v_{21} = 2u'_{21}.$$

The first equation shows that the tangent to the curve C_σ at P_σ intersects the generator g of S. Denote this point of intersection by $P_{\sigma'}$. The locus of the point $P_{\sigma'}$ is therefore a curve on S, $C_{\sigma'}$. Its tangent at $P_{\sigma'}$ is obtained by joining $P_{\sigma'}$ to the point $P_{\sigma''}$ defined by the second equation (4). But $P_{\sigma''}$ is in the plane $P_z P_\rho P_\sigma$ if and only if $v_{21} = 0$. Put

$$\tau = u_{21}\rho + u_{11}\sigma,$$

then P_τ is a point on $P_\rho P_\sigma$ such that the line joining it to $P_{\sigma'}$ is a generator of the hyperboloid H osculating S along g. We see that the tangent to $C_{\sigma'}$ intersects $P_z P_\tau$ if and only if $v_{21} = 0$.

The principal surface of the congruence Γ is therefore defined by the following theorem:

We draw upon the ruled surface S any asymptotic line C_z, and upon any surface S' of the congruence Γ the curve C_σ which corresponds to it, P_z and P_σ being corresponding points of the two curves. The tangent to C_σ at P_σ always intersects the generator g of S, which passes through P_z, in a certain point $P_{\sigma'}$, whose locus gives a curve $C_{\sigma'}$ upon S. Upon S' a point P_τ is constructed such that the line $P_\tau P_{\sigma'}$ shall be a generator of the hyperboloid osculating S along g. The tangent to $C_{\sigma'}$ at $P_{\sigma'}$ intersects the line $P_z P_\tau$ if and only if the surface S' is the principal surface of the congruence.

§ 2. *The derivative conic.*

In the general case where $\theta_4 \neq 0$, the hyperboloid H osculating S along g and the hyperboloid H' osculationg S' along g' intersect along g' and a space cubic which we have called the derivative cubic. Moreover this cubic does not degenerate unless either S has a straight line directrix or S' is developable.

In the present case however, H and H' have besides g' the straight line f in common, i. e., the flecnode tangent passing through P_y. The rest of their intersection is therefore a conic which we shall call the *derivative conic*.

We proceed to prove these statements and to derive the equations of the derivative conic. Taking as tetrahedron of reference the tetrahedron $P_y P_z P_\rho P_\sigma$, the equation of H is

$$x_1 x_4 - x_2 x_3 = 0,$$

and that of H'

$$(u_{22}x_1 - u_{21}x_2)\left[Jx_4 - (\lambda_{12}u_{22} - \lambda_{22}u_{12})x_1 + (\lambda_{12}u_{21} - \lambda_{22}u_{11})x_2\right]$$
$$+ (u_{12}x_1 - u_{11}x_2)\left[Jx_3 - (\lambda_{11}u_{22} - \lambda_{21}u_{12})x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2\right] = 0,$$

as we have shown in *Studies*, equations (1) and (3).

In our case we may put

$$u_{12} = u_{11} - u_{22} = 0, \qquad J = u_{11}^2$$

whence

(5) $\qquad v_{11} - v_{22} = 2p_{12}u_{21}, \qquad v_{12} = 0, \qquad v_{21} = 2u_{21}' - (p_{11} - p_{22})u_{21}$

and

(6)
$$2J\lambda_{11} = -u_{11}v_{11}, \qquad\qquad 2J\lambda_{12} = 0,$$
$$2J\lambda_{21} = u_{21}v_{22} - u_{11}v_{21}, \qquad 2J\lambda_{22} = -u_{11}v_{22}.$$

We find, therefore, for H' the equation

$$(u_{11}x_1 - u_{21}x_2)\left[Jx_4 - \lambda_{22}u_{11}x_2\right] - u_{11}x_2\left[Jx_3 - \lambda_{11}u_{11}x_1 + (\lambda_{11}u_{21} - \lambda_{21}u_{11})x_2\right] = 0,$$

or

(7) $\quad Ju_{11}(x_1x_4 - x_2x_3) + u_{11}^2(\lambda_{11} - \lambda_{22})x_1x_2 - Ju_{21}x_2x_4 + [u_{11}^2\lambda_{21} - u_{11}u_{21}(\lambda_{11} - \lambda_{22})]x_2^2 = 0,$

while the equation of H is

$$x_1x_4 - x_2x_3 = 0.$$

Both equations are satisfied by $x_1 = x_2 = 0$, as well as by $x_2 = x_4 = 0$ which proves that g' and f are lines upon both of these hyperboloids. They must therefore have also a conic in common, whose plane must, according to (7), have the equation

$$u_{11}(\lambda_{11} - \lambda_{22})x_1 - u_{11}u_{21}x_4 + \left[u_{11}\lambda_{21} - u_{21}(\lambda_{11} - \lambda_{22})\right]x_2 = 0.$$

If we put for abbreviation

(8) $\quad A = u_{11}(\lambda_{11} - \lambda_{22}), \qquad B = -u_{11}\lambda_{21} + u_{21}(\lambda_{11} - \lambda_{22}), \qquad C = u_{11}u_{21},$

whence

(9) $\qquad\qquad\qquad\qquad\qquad u_{11}A + p_{12}C = 0,$

we have therefore, as the *equations of the derivative conic*

(10) $\qquad\qquad Ax_1 - Bx_2 - Cx_4 = 0, \qquad x_1x_4 - x_2x_3 = 0.$

We can also express the coördinates of any point on the conic in terms of a parameter t. Any point on the hyperboloid H can be represented in the form

$$x_1 = ut, \qquad x_2 = t, \qquad x_3 = u, \qquad x_4 = 1.$$

This point is, moreover, a point of the conic if the condition

$$Aut - Bt - C = 0$$

is satisfied, whence

$$u = \frac{C + Bt}{At}.$$

If we substitute into the above equations for $x_1 \cdots x_4$ and multiply by At, we find

$$(11) \qquad x_1 = t(C + Bt), \qquad x_2 = At^2, \qquad x_3 = C + Bt, \qquad x_4 = At$$

as the parametric equations of the conic, or in homogeneous form

$$(12) \qquad x_1 = t_1 \psi, \qquad x_2 = At_1^2, \qquad x_3 = t_2 \psi, \qquad x_4 = At_1 t_2,$$

where

$$(13) \qquad\qquad\qquad \psi = Bt_1 + Ct_2.$$

The conic of course always passes through P_ρ. The first question which we naturally ask is this: when does the conic degenerate into a pair of lines? Clearly this can only happen if the plane

$$Ax_1 - Bx_2 - Cx_4 = 0$$

intersects the hyperboloid H in a pair of lines, i. e., if it is tangent to H. Moreover since this plane contains P_ρ, it must in that case contain at least one of the two generators of H which pass through P_ρ. If it contains that one which passes also through P_σ we must have $C = 0$, i. e., since $u_{21} \neq 0$ (S not being a quadric), $u_{11} = 0$. S' must therefore be a developable. If however this plane contains the other generator through P_ρ, namely that one which passes through P_y, we must have $A = 0$, which gives either $u_{11} = 0$ as before or else $p_{12} = 0$, in which case the flecnode curve C_y would be a straight line.

  Therefore, the derivative conics degenerate if and only if the surface S has a straight line directrix, or else if the derivative of S with respect to x is one of the developable surfaces of the congruence Γ.

  By an investigation similar to that in *Studies*, pp. 230–232, we obtain the further result:

  Two consecutive derivative conics never intersect unless they degenerate.

  § 3. *The developable surface generated by the plane of the derivative conic.*

  As x changes, the plane of the conic C_x envelops a developable surface, the equations of whose generator we shall now proceed to determine.

  Let us form $\partial \phi / \partial x$ under the assumption that $t_1 : t_2$ is independent of x. Then

$$\phi + \frac{\partial \phi}{\partial x} \, \delta x$$

will represent any point on the derivative conic $C_{x+\delta x}$ belonging to the argument $x + \delta x$, where δx is an infinitesimal. The plane of this conic will be determined by any three points upon it. We have (again assuming $p_{21} = 0$),

$$\frac{\partial \phi}{\partial x} = y\left[t_1(B't_1 + C't_2) + \tfrac{1}{2}(Bt_1 + Ct_2)u_{11}t_2 + \tfrac{1}{2}At_1t_2u_{21}\right]$$

$$+ z\left[A't_1^2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_1 + \tfrac{1}{2}At_1t_2u_{11}\right]$$

(14)

$$+ \rho\left[t_2(B't_1 + C't_2) + \tfrac{1}{2}(Bt_1 + Ct_2)t_1\right]$$

$$+ \sigma\left[A't_1t_2 - \tfrac{1}{2}(Bt_1 + Ct_2)p_{12}t_2 + \tfrac{1}{2}At_1^2\right].$$

We can obtain three points of the conic $C_{x+\delta x}$ by putting $t_1 = 0$, $t_2 = 1$; $t_1 = 1$, $t_2 = 0$; $t_1 = + C$, $t_2 = - B$, which last set of values corresponds to $\psi = 0$. Therefore the equation of the plane of $C_{x+\delta x}$ is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ Cu_{11}\delta x & 0 & 2C + 2C'\delta x & -Cp_{12}\delta x \\ 2B + 2B'\delta x & 2A + (2A' - Bp_{12})\delta x & B\delta x & A\delta x \\ [2C(B'C - BC') & 2AC^2 + (2A'C^2 & -2B(B'C & -2ABC + (AC^2 \\ -ABCu_{21}]\delta x & -ABCu_{11})\delta x & -BC')\delta x & -2A'BC)\delta x \end{vmatrix} = 0.$$

If this determinant be developed, retaining of course only the terms of the first order in δx, we find that the planes of C_x and of $C_{x+\delta x}$ intersect along the line

(15)
$$Ax_1 - Bx_2 - Cx_4 = 0,$$
$$ADx_1 - Ex_2 - CFx_4 = 0,$$

where

$$D = 4A'C + 2AC' - BCp_{12},$$

(16) $$E = 2BCA' + 2ACB' + 2ABC' - AC^2,$$

$$F = 4AC' + 2A'C - ABu_{11} + A^2u_{21} = 4AC' + 2A'C + Au_{11}^2\lambda_{21}.$$

Equations (15) are therefore the equations of the generators of the developable surface.

This proof would not be valid if either C or B were zero. For then the third of the three points of the conic $C_{x+\delta x}$ which we have used to determine its plane would coincide with one of the other two. If $C = 0$ the conic degenerates. Since however we might in the case $B = 0$ choose three other points of $C_{x+\delta x}$, as we might also do in the general case, the result will obviously be obtained from the general case by substituting $B = 0$.

We see from (15) that the generator of the developable surface passes through P_ρ and that it intersects the plane $x_3 = 0$, in the point P_χ or

$$[C(E - BF), \; AC(D - F), \; 0, \; A(E - BD)],$$

which we may therefore represent by the expression

$$\text{(17)} \qquad \chi = C(E - BF)y + AC(D - F)z + A(E - BD)\sigma.$$

Since the surface generated by this line joining P_ρ to P_x is developable, it must be possible to find four functions $\alpha, \beta, \gamma, \delta$ of x such that

$$\text{(18)} \qquad \alpha\rho + \beta\chi + \gamma\rho' + \delta\chi' = 0.$$

Now we have

$$\rho' = \tfrac{1}{2}(u_{11}y - p_{12}\sigma),$$

and we find

$$\text{(19)} \qquad \chi' = Gy + Hz + M\rho + N\sigma,$$

where

$$G = C(E' - BF' - B'F) + C'(E - BF) + \tfrac{1}{2}u_{21}A(E - BD),$$
$$H = (AC' + A'C)(D - F) + AC(D' - F') - \tfrac{1}{2}p_{12}C(E - BF)$$
$$\text{(20)} \qquad\qquad\qquad\qquad\qquad\qquad + \tfrac{1}{2}u_{11}A(E - BD),$$
$$M = \tfrac{1}{2}C(E - BF),$$
$$N = A'(E - BD) + A(E' - BD' - B'D) + \tfrac{1}{2}AC(D - F).$$

If we substitute these values of χ' and ρ' and also the expression (17) for χ into (18), we find that $\alpha, \beta, \gamma, \delta$ must satisfy the equations

$$\text{(21)} \qquad \begin{aligned} C(E - BF)\beta + \tfrac{1}{2}u_{11}\gamma + G\delta &= 0, \\ AC(D - F)\beta \qquad\qquad\;\; + H\delta &= 0, \\ A(E - BD)\beta - \tfrac{1}{2}p_{12}\gamma + N\delta &= 0, \\ \alpha + M\delta &= 0. \end{aligned}$$

Therefore the determinant of the first three equations, which expanded becomes

$$\text{(22)} \qquad \tfrac{1}{2}p_{12}C(D - F)[NC - GA + HB],$$

must vanish identically; i. e., since the other factors do not vanish identically, we must have

$$\text{(23)} \qquad NC - GA + HB = 0.$$

We may also verify (23) directly. For we find from (20),

$$\begin{aligned} NC - GA + HB = {}& F(ACB' - \tfrac{1}{2}AC^2 - BA'C + \tfrac{1}{2}B^2Cp_{12}) \\ & - D(ACB' - \tfrac{1}{2}AC^2 - BAC' - \tfrac{1}{2}A^2Bu_{21} + \tfrac{1}{2}AB^2u_{11}) \\ & + E(A'C - AC' - \tfrac{1}{2}A^2u_{21} - \tfrac{1}{2}CBp_{12} + \tfrac{1}{2}ABu_{11}) \\ = {}& \tfrac{1}{2}[F(E - BD) - D(E - BF) + E(D - F)] = 0. \end{aligned}$$

We can now determine the edge of regression of the developable surface. If $\gamma\rho + \delta\chi$ is a point on this curve, its tangent constructed at that point must coincide with the generator of the developable, i. e.,

(24) $$\gamma'\rho + \delta'\chi + \gamma\rho' + \delta\chi' = \lambda\chi + \mu\rho$$

or

$$(\gamma' - \mu)\rho + (\delta' - \lambda)\chi + \gamma\rho' + \delta\chi' = 0,$$

which is identical with (18) if we put there

$$\alpha = \gamma' - \mu, \qquad \beta = \delta' - \lambda.$$

But on account of (23) we can determine α, β, γ, δ so as to satisfy (18); we can therefore determine λ, μ, γ, δ so as to satisfy (24). Moreover we find

(25) $$\gamma : \delta = -2GA(D - F) + 2H(E - BF) : u_{11}A(D - F).$$

Therefore the edge of regression of the developable is given by the expression

(26)
$$\kappa = [-2GA(D - F) + 2H(E - BF)]\rho$$
$$+ u_{11}A(D-F)[C(E-BF)y + AC(D-F)z + A(E-BD)\sigma].$$

We see from (15) that the generator of the developable surface coincides with one of the generators of H which passes through P_ρ, only if either A or C vanishes, i. e., either if S' is developable or if S has a straight line directrix, in which cases the derivative conic degenerates. *The generator of the developable is tangent to H at P_ρ, neglecting the cases just mentioned only if $D - F = 0$.* As (26) shows, *the cuspidal edge of the developable then coincides with C_ρ.* If the expressions for A, B, C be substituted into the condition $D - F = 0$, or

$$2(A'C - AC') - BCp_{12} + ABu_{11} - A^2u_{21} = 0,$$

it becomes

$$4\frac{u'_{11}}{u_{11}} - 2\frac{p'_{12}}{p_{12}} - 2\frac{u'_{21}}{u_{21}} = 0,$$

which gives on integration

(27) $$\frac{u_{11}^4}{p_{12}^2 u_{21}^2} = \text{const.}$$

If $E - BF = 0$, P_κ lies in the plane $P_\rho P_z P_\sigma$, and if $E - BD = 0$ in the plane $P_y P_z P_\rho$.

It will clearly be possible to characterize special classes of ruled surfaces (for $\theta_4 = 0$) by special properties of the developable surfaces here considered. As we are here dealing with the general theory of such surfaces only, we will not go into such details at present.

We will also merely mention the fact that the relation of S to its flecnode surface F is especially close in this case ($\theta_4 = 0$). In fact S is also the flecnode surface of F. Moreover, the same hyperboloid H which osculates S along g, also osculates F along the corresponding generators f of F. The con-

gruence Γ' which belongs to the surface F' in the same way as Γ does to S, is therefore made up of the generators of the second set on the osculating hyperboloids of S, those of the first set constituting the lines of the congruence Γ All of these remarks follow easily from the equations of the flecnode surface, which, under the assumptions $p_{11} = p_{22} = p_{21} = 0$, assume the form

(28)
$$y'' - 2\frac{q_{12}}{p_{12}}y' - \rho' - q_{11}y + \frac{q_{12}}{p_{12}}\rho = 0,$$

$$\rho'' + 4q_{11}y' - 2\frac{q_{12}}{p_{12}}\rho' + \left[2q'_{11} - p_{12}q_{21} - 4q_{11}\frac{q_{12}}{p_{12}}\right]y - q_{11}\rho = 0.$$

Moreover, since S' is a developable surface of the congruence Γ if $q_{11} = 0$, and since (28) shows that C_ρ is then an asymptotic curve on F', we see that the developable surfaces of the congruence intersect its focal surface F' along asymptotic lines, as it should according to the general theory of congruences.

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